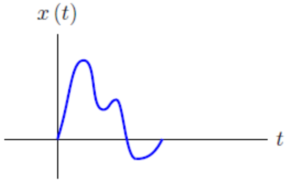
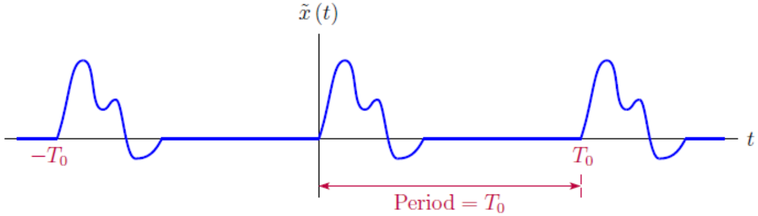
4.3 Analysis of Non-Periodic Continuous-Time Signals

Fourier transform

A non-periodic signal $x(t)$:



Periodic extension $\tilde{x}(t)$ of the signal $x(t)$:



$$\tilde{x}(t) = \ldots + x(t + T_0) + x(t) + x(t - T_0) + x(t - 2T_0) + \ldots$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kT_0)$$

If the period T_0 is allowed to become very large, the periodic signal $\tilde{x}(t)$ would start to look more and more similar to $x(t)$. In the limit we would have

$$\lim_{T_0 \to \infty} [\tilde{x}(t)] = x(t) \qquad (4.117)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Fourier transform (continued)

Fourier transform for continuous-time signals

Synthesis equation: (Inverse transform)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Analysis equation: (Forward transform)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Shorthand notation:

$$X(\omega) = \mathcal{F}\{x(t)\}\ , \qquad x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Fourier transform (continued)

Fourier transform for continuous-time signals (using f instead of ω)

Synthesis equation: (Inverse transform)

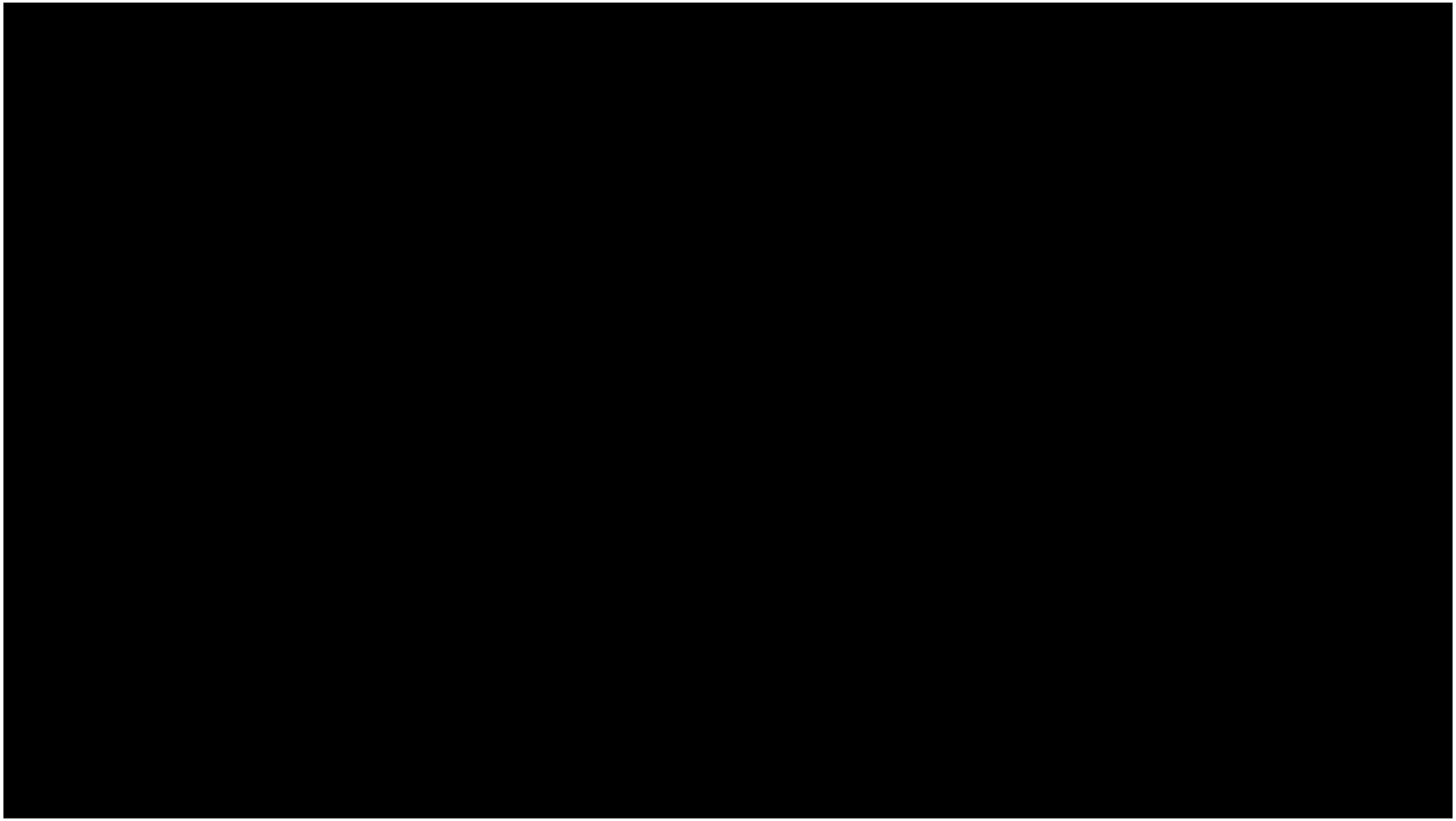
$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

Analysis equation: (Forward transform)

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Note the lack of the scale factor $1/2\pi$ in front of the integral of the inverse transform when f is used. This is consistent with the relationship $d\omega = 2\pi\, df$.

4.3 Analysis of Non-Periodic Continuous-Time Signals



[But what is the Fourier Transform? A visual introduction. - YouTube]

4.3 Analysis of Non-Periodic Continuous-Time Signals

Existence of Fourier transform

Let $\hat{x}(t)$ be defined as

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\varepsilon(t) = x(t) - \hat{x}(t) = x(t) - \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

For perfect convergence we want $\varepsilon(t) = 0$ for all t. However, this is not possible at time instants for which $x(t)$ exhibits discontinuities.

Dirichlet conditions for existence of the Fourier transform

- The signal $x(t)$ must be integrable in an absolute sense:

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

- If the signal $x(t)$ has discontinuities, it must have at most a finite number of them in any finite time interval.
- The signal $x(t)$ must have at most a finite number of minima and maxima in any finite time-interval.

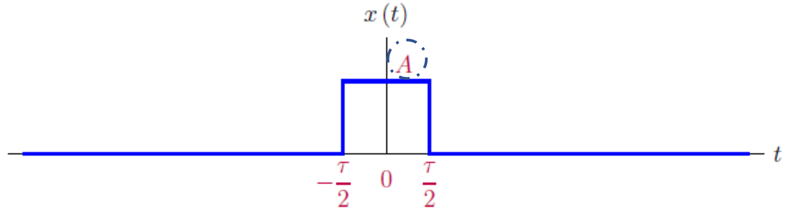
4.3 Analysis of Non-Periodic Continuous-Time Signals

Developing further insight

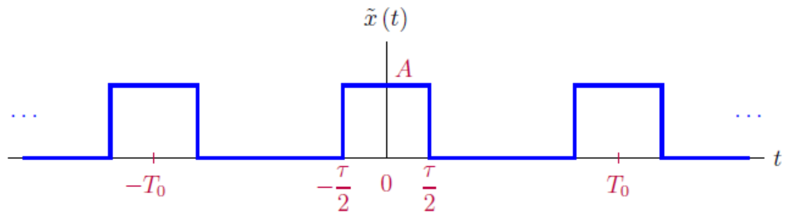
Example 4.7

$$c_k = d \operatorname{sinc}(kd)$$

An isolated rectangular pulse $x(t) = A\,\Pi(t/\tau)$:



Periodic extension of $x(t)$ into a pulse train:



EFS coefficients of $\tilde{x}(t)$:

$$c_k = Ad \operatorname{sinc}(kd)$$

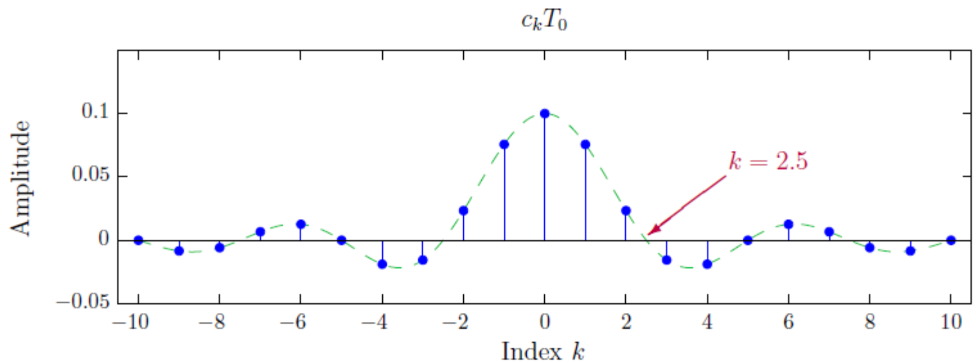
$$\text{Duty cycle: } d = \frac{\tau}{T_0} \quad \Rightarrow \quad c_k = \frac{A\tau}{T_0} \operatorname{sinc}(k\tau/T_0)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

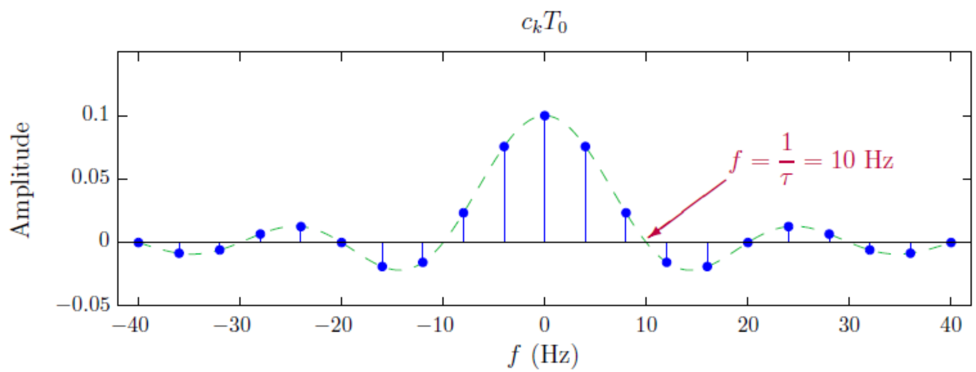
Developing further insight (continued)

Multiply both sides by T_0: $c_k T_0 = A\tau\ \text{sinc}\,(k f_0 \tau)$

Graph coefficients $c_k T_0$ for $A = 1$, $\tau = 0.1$ seconds, $T_0 = 0.25$ seconds, $f_0 = 4$ Hz:



Use actual frequencies in Hz on the horizontal axis:



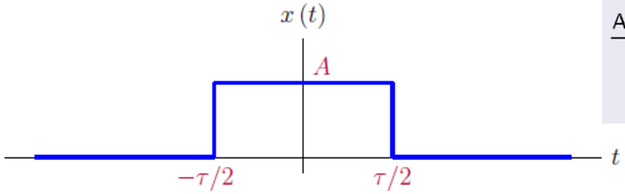
4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.12

Fourier transform of a rectangular pulse

Using the forward Fourier transform integral, find the Fourier transform of the isolated rectangular pulse signal

$$x(t) = A\,\Pi\left(\frac{t}{\tau}\right)$$



Analysis equation: (Forward transform)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\tilde{x}(t) = A \sin(\omega_0 t + \theta)$$
$$= \frac{A}{2j} e^{j(\omega_0 t + \theta)} - \frac{A}{2j} e^{-j(\omega_0 t + \theta)}$$

Solution:

Use Fourier transform integral:

$$X(\omega) = \int_{-\tau/2}^{\tau/2} (A)\, e^{-j\omega t}\, dt = A\, \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-\tau/2}^{\tau/2} = \frac{2A}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$$

Use the sinc function:

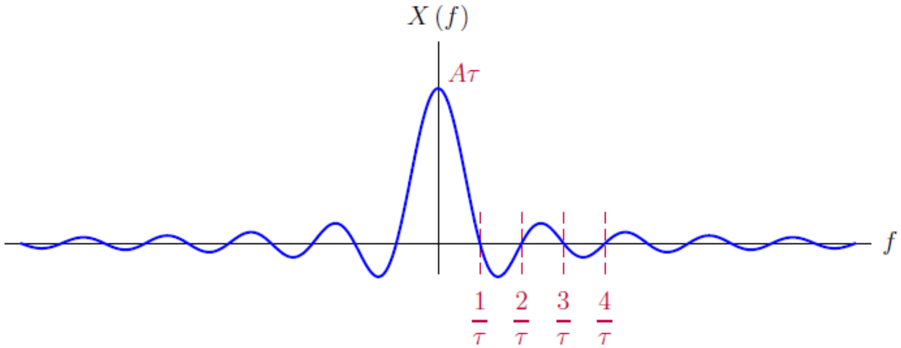
$$X(\omega) = A\tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = A\tau\, \text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.12 (continued)

Substitute $\omega = 2\pi f$:

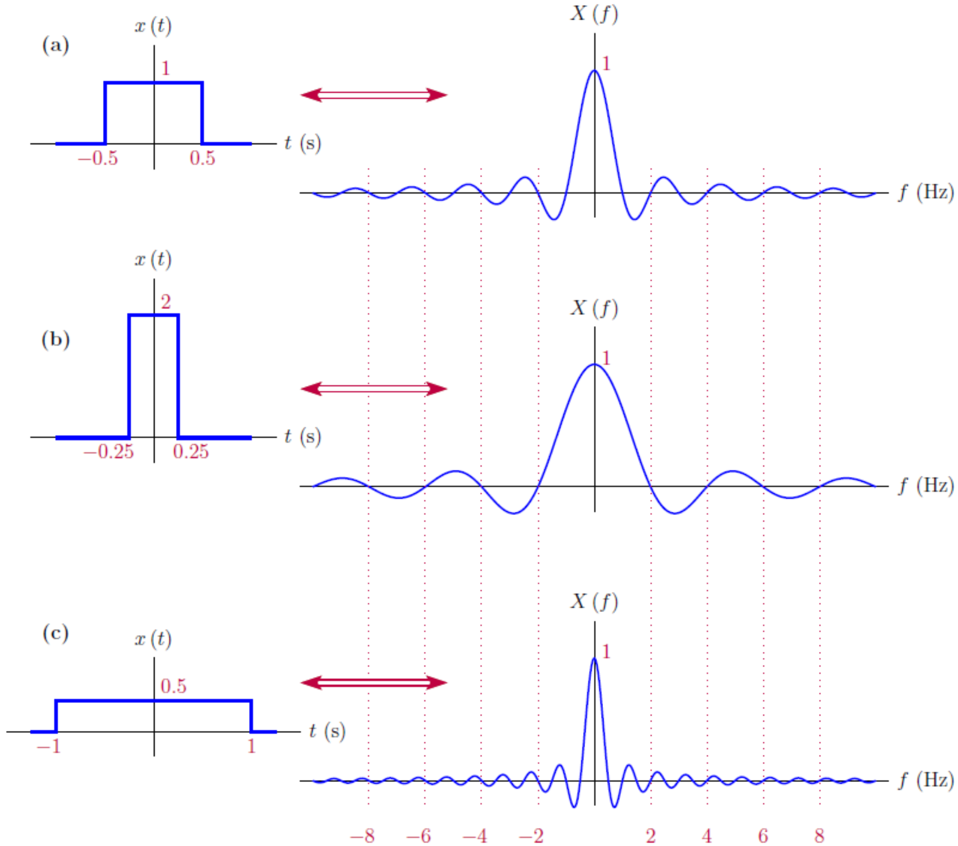
$$X(f) = A\tau \operatorname{sinc}(f\tau)$$



The peak value of the spectrum is $A\tau$, and occurs at the frequency $f = 0$.
The zero crossings of the spectrum occur at frequencies that satisfy $f\tau = k$, where k is any non-zero integer.

4.3 Analysis of Non-Periodic Continuous-Time Signals

Fourier transform of isolated pulse



4.3 Analysis of Non-Periodic Continuous-Time Signals

Fourier transform of isolated pulse

Observations on the transform of isolated pulse:

- Largest values of the spectrum occur at frequencies close to $f = 0$. Thus, low frequencies are more significant in the spectrum, and the significance of frequency components decreases as we move further away from $f = 0$ in either direction.
- The zero-crossings of the spectrum occur for values of f that are integer multiples of $1/\tau$. If the pulse width is decreased, zero-crossings move further away from the frequency $f = 0$ resulting in the spectrum being *stretched out* in both directions. This increases the relative significance of large frequencies. Narrower pulses have frequency spectra that expand to higher frequencies.
- If the pulse width is increased, zero-crossings of the spectrum move inward, closer to the frequency $f = 0$ resulting in the spectrum being squeezed in from both directions. This decreases the significance of large frequencies, and causes the spectrum to be concentrated more heavily around the frequency $f = 0$. Wider pulses have frequency spectra that are more concentrated at low frequencies.

4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.14

Transform of the unit-impulse function

Find the Fourier transform of the unit-impulse function.

<u>Solution:</u>

$$\mathcal{F}\{\,\delta\,(t)\,\} = \int_{-\infty}^{\infty} \delta\,(t)\; e^{-j\omega t}\, dt = e^{-j\omega t}\,\Big|_{t=0} = 1$$

Alternative approach:

$$q\,(t) = \frac{1}{a}\,\Pi\left(\frac{t}{a}\right) \qquad \text{(Pulse with unit area)}$$

$$Q\,(f) = \mathcal{F}\{\,q(t)\,\} = \mathrm{sinc}\,(\,f a\,)$$

Express the unit-impulse function using $q\,(t)$:

$$\delta\,(t) = \lim_{a\to 0}\,\{q\,(t)\} \qquad \Rightarrow \qquad \mathcal{F}\{\,\delta\,(t)\,\} = \lim_{a\to 0}\,\{\,Q\,(f)\,\} = \lim_{a\to 0}\,\{\,\mathrm{sinc}\,(\,f a\,)\,\} = 1$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

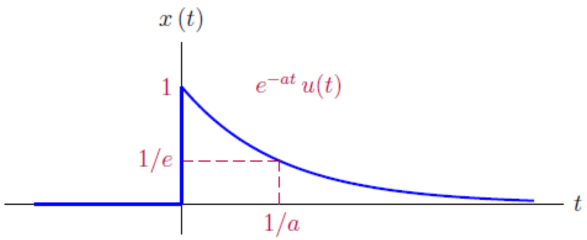
Example 4.15

Fourier transform of a right-sided exponential signal

Determine the Fourier transform of the right-sided exponential signal

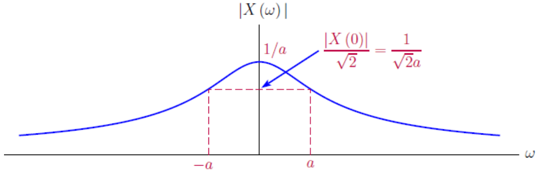
$$x\left(t\right)=e^{-at}\,u\left(t\right)$$

with $a>0$.



Solution:

$$X\left(\omega\right)=\int_0^{\infty}e^{-at}\,e^{-j\omega t}\,dt=\int_0^{\infty}e^{-(a+j\omega)t}\,dt=\frac{1}{a+j\omega}$$



4.3 Analysis of Non-Periodic Continuous-Time Signals

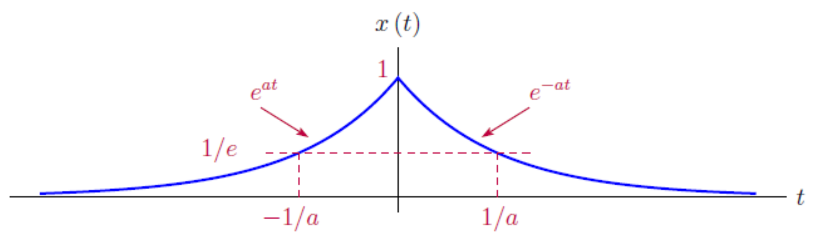
Example 4.16

Fourier transform of a two-sided exponential signal

Determine the Fourier transform of the two-sided exponential signal given by

$$x(t) = e^{-a|t|}$$

where a is any non-negative real-valued constant.



<u>Solution:</u>

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

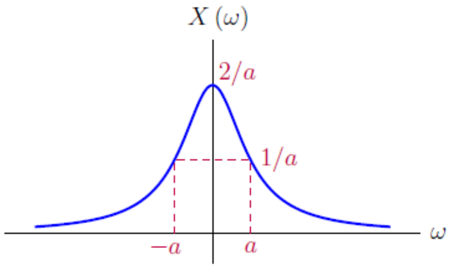
Split the integral into two halves:

$$X(\omega) = \int_{-\infty}^{0} e^{-a|t|}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt = \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.16 (continued)

$$X(\omega) = \frac{2a}{a^2 + \omega^2}$$



4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform

Linearity:

Consider any two signals $x_1(t)$ and $x_2(t)$ with their respective transforms:

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \qquad \text{and} \qquad x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

Linearity of the Fourier transform

For any two constants α_1 and α_2:

$$\alpha_1\, x_1(t) + \alpha_2\, x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha_1\, X_1(\omega) + \alpha_2\, X_2(\omega)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Duality:

Swap the roles of the signal and the transform:

Duality property

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \text{implies that} \quad X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$$

It is more convenient to express the duality property using the frequency f instead of the radian frequency ω:

Duality property (using f instead of ω)

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) \quad \text{implies that} \quad X(t) \overset{\mathcal{F}}{\longleftrightarrow} x(-f)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.19

Fourier transform of the sinc function

Find the Fourier transform of the signal

$$x(t) = \mathrm{sinc}(t)$$

Solution:

Recall that the Fourier transform of a rectangular pulse was found in Example 4.12:

$$\mathcal{F}\left\{ A\,\Pi\left(\frac{t}{\tau}\right) \right\} = A\tau\ \mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Let $\tau = 2\pi$ so that the argument of the sinc function becomes ω:

$$\mathcal{F}\left\{ A\,\Pi\left(\frac{t}{2\pi}\right) \right\} = 2\pi\,A\ \mathrm{sinc}(\omega)$$

Let $A = 1/2\pi$:

$$\mathcal{F}\left\{ \frac{1}{2\pi}\,\Pi\left(\frac{t}{2\pi}\right) \right\} = \mathrm{sinc}(\omega)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

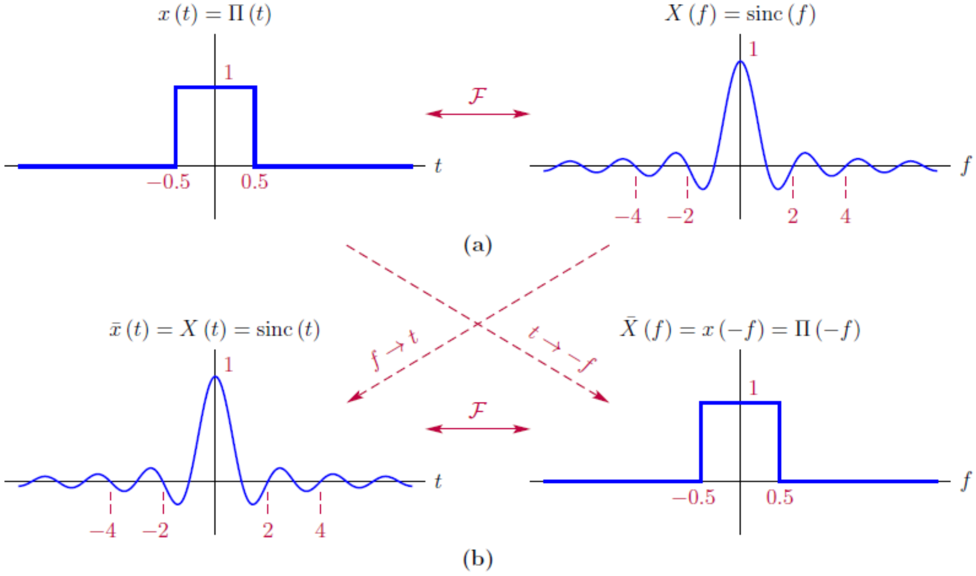
Example 4.19 (continued)

Apply the duality property:

$$\mathcal{F}\left\{\operatorname{sinc}(t)\right\} = \Pi\left(\frac{-\omega}{2\pi}\right) = \Pi\left(\frac{\omega}{2\pi}\right)$$

Using f instead of ω yields a result that is easier to remember:

$$\mathcal{F}\left\{\operatorname{sinc}(t)\right\} = \Pi(f)$$



4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.20

Transform of a constant-amplitude signal

Find the Fourier transform of the constant-amplitude signal

$$x(t) = 1, \quad \text{all } t$$

<u>Solution:</u>

Direct application of the Fourier transform integral does not work:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} (1)\, e^{-j\omega t}\, dt$$

Recall that, in Example 4.14, the Fourier transform of the unit-impulse signal was found to be a constant for all frequencies:

$$\mathcal{F}\{\delta(t)\} = 1\,, \quad \text{all } \omega$$

Apply the duality property:

$$\mathcal{F}\{1\} = 2\pi\,\delta(-\omega) = 2\pi\,\delta(\omega)$$

Using f instead of ω:

$$\mathcal{F}\{1\} = \delta(f)$$

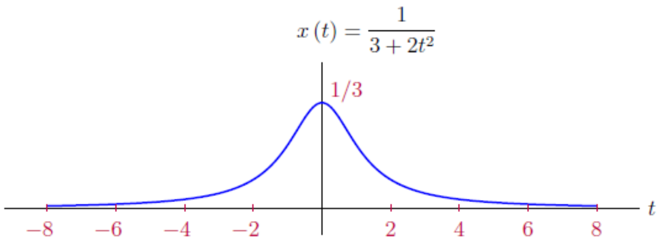
4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.21

Another example of using the duality property

Find the Fourier transform of the signal

$$x(t) = \frac{1}{3 + 2t^2}$$



Solution:

Recall that, in Example 4.16 we have found

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

Apply duality property

$$\frac{2a}{a^2 + t^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, e^{-a|\omega|}$$

Multiply both the numerator and the denominator of the time-domain component by 2:

$$\frac{4a}{2a^2 + 2t^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, e^{-a|\omega|}$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.21 (continued)

Choose

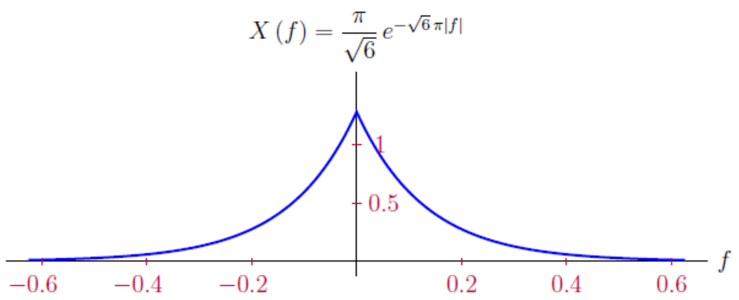
$$2a^2 = 3 \quad \Longrightarrow \quad a = \sqrt{\frac{3}{2}}$$

so that

$$\frac{2\sqrt{6}}{3+2t^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, e^{-\sqrt{\frac{3}{2}}\,|\omega|}$$

Scale both sides with $2\sqrt{6}$:

$$\frac{1}{3+2t^2} \overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{\sqrt{6}}\, e^{-\sqrt{\frac{3}{2}}\,|\omega|}$$



4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Symmetry of the Fourier transform:

A transform $X(\omega)$ is said to be *conjugate symmetric* if it satisfies

$$X^*(\omega) = X(-\omega) \qquad \text{for all } \omega$$

A transform $X(\omega)$ is said to be *conjugate antisymmetric* if it satisfies

$$X^*(\omega) = -X(-\omega) \qquad \text{for all } \omega$$

Symmetry of the Fourier transform

$$x(t)\text{: Real, } \operatorname{Im}\{x(t)\} = 0 \quad \text{implies that} \quad X^*(\omega) = X(-\omega)$$

$$x(t)\text{: Imag, } \operatorname{Re}\{x(t)\} = 0 \quad \text{implies that} \quad X^*(\omega) = -X(-\omega)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.24

Symmetry properties for the transform of right-sided exponential signal

The Fourier transform of the right-sided exponential signal $x(t) = e^{-at}\,u(t)$ was found in Example 4.15 to be

$$X(f) = \mathcal{F}\left\{e^{-at}\,u(t)\right\} = \frac{1}{a+j\omega}$$

Elaborate on the symmetry properties of the transform.

Solution:

Since $x(t)$ is real-valued, its transform must be conjugate symmetric:

$$X^*(\omega) = \left(\frac{1}{a+j\omega}\right)^* = \frac{1}{a-j\omega}\,, \qquad X(-\omega) = \left(\frac{1}{a+j\omega}\right)\Bigg|_{\omega\to-\omega} = \frac{1}{a-j\omega}$$

It follows that

$$X^*(\omega) = X(-\omega) \qquad \Rightarrow \qquad X(\omega) \text{ is conjugate symmetric.}$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Transforms of even and odd signals:

Real valued signal with even symmetry

$$x(-t) = x(t) \ , \text{all } t \quad \text{implies that} \quad \text{Im}\{X(\omega)\} = 0 \ , \text{all } \omega$$

Real valued signal with odd symmetry

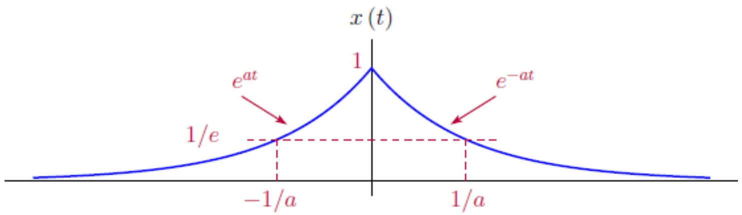
$$x(-t) = -x(t) \ , \text{all } t \quad \text{implies that} \quad \text{Re}\{X(\omega)\} = 0 \ , \text{all } \omega$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.25

Transform of a two-sided exponential signal

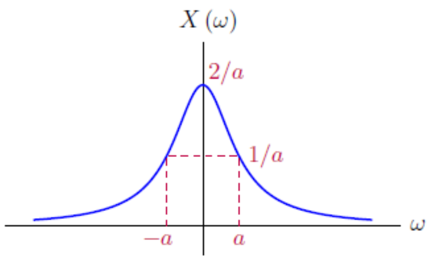
Elaborate on the symmetry properties of the Fourier transform of the exponential signal $x(t) = e^{-a|t|}$.



Solution:

The Fourier transform is

$$X(\omega) = \frac{2a}{a^2 + \omega^2}$$



$x(t)$ is real $\Rightarrow$ $X(\omega)$ is conjugate symmetric.

$x(t)$ is even $\Rightarrow$ $X(\omega)$ is real.

Since $X(\omega)$ is both conjugate symmetric and real $\Rightarrow$ $X(\omega)$ is even.

4.3 Analysis of Non-Periodic Continuous-Time Signals

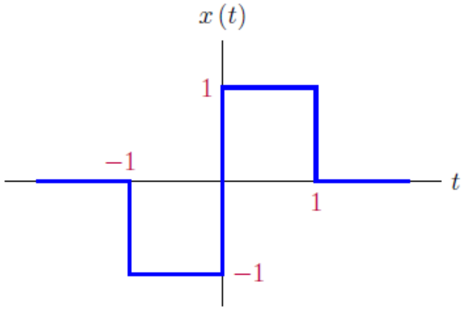
Example 4.26

Transform of a pulse with odd symmetry

Determine the Fourier transform of the signal

$$x(t) = \begin{cases} -1\,, & -1 < t < 0 \\ 1\,, & 0 < t < 1 \\ 0\,, & t < -1 \quad \text{or} \quad t > 1 \end{cases}$$

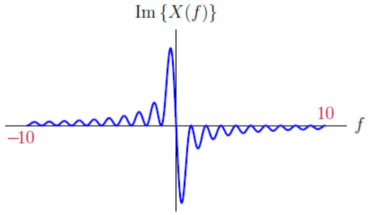
and show that it is purely imaginary.



Solution:

Through direct use of the forward transform integral:

$$X(\omega) = \int_{-1}^{0} (-1)\, e^{-j\omega t}\, dt + \int_{0}^{1} (1)\, e^{-j\omega t}\, dt = \frac{j2}{\omega} \left[\cos(\omega) - 1 \right]$$



4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Time shifting:

Time shifting property

For a transform pair

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

it can be shown that

$$x(t-\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)\, e^{-j\omega\tau}$$

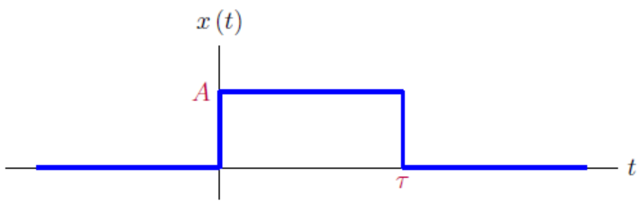
4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.27

Time shifting a rectangular pulse

Using the time shifting property, find the transform of the isolated rectangular pulse given by

$$x(t) = A\,\Pi\left(\frac{t-\tau/2}{\tau}\right)$$



Solution:

The transform of a rectangular pulse with amplitude A, width τ and center at $t=0$ was found in Example 4.12:

$$A\,\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\longleftrightarrow} A\tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Use time-shifting property:

$$A\,\Pi\left(\frac{t-\tau/2}{\tau}\right) \overset{\mathcal{F}}{\longleftrightarrow} A\tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)\,e^{-j\omega\tau/2}$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

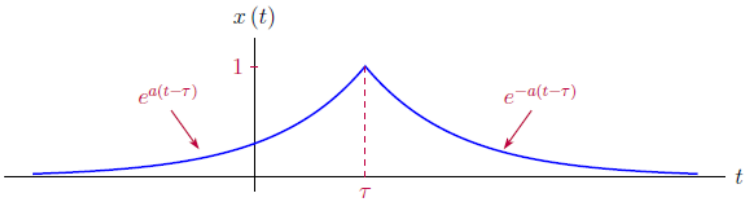
Example 4.28

Time shifting a two-sided exponential signal

Determine the Fourier transform of the signal

$$x(t) = e^{-a\,|t-\tau\,|}$$

where $a > 0$.



Solution:

In Example 4.16 it was determined that

$$\mathcal{F}\left\{ e^{-a\,|t\,|} \right\} = \frac{2a}{a^2+\omega^2}$$

Use the time shifting property:

$$X(\omega) = \mathcal{F}\left\{ e^{-a\,|t-\tau\,|} \right\} = \frac{2a\,e^{-j\omega\tau}}{a^2+\omega^2}$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Frequency shifting:

Frequency shifting property

For a transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

it can be shown that

$$x(t)\, e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_0)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Modulation property:

Modulation property is an interesting consequence of the frequency shifting property combined with the linearity of the Fourier transform.

Modulation property

For a transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

it can be shown that

$$x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

and

$$x(t)\sin(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(\omega-\omega_0)\,e^{-j\pi/2}+X(\omega+\omega_0)\,e^{j\pi/2}\right]$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.29

Modulated pulse

Find the Fourier transform of the modulated pulse given by

$$x(t) = \begin{cases} \cos(2\pi f_0 t), & |t| < \tau \\ 0, & |t| > \tau \end{cases}$$

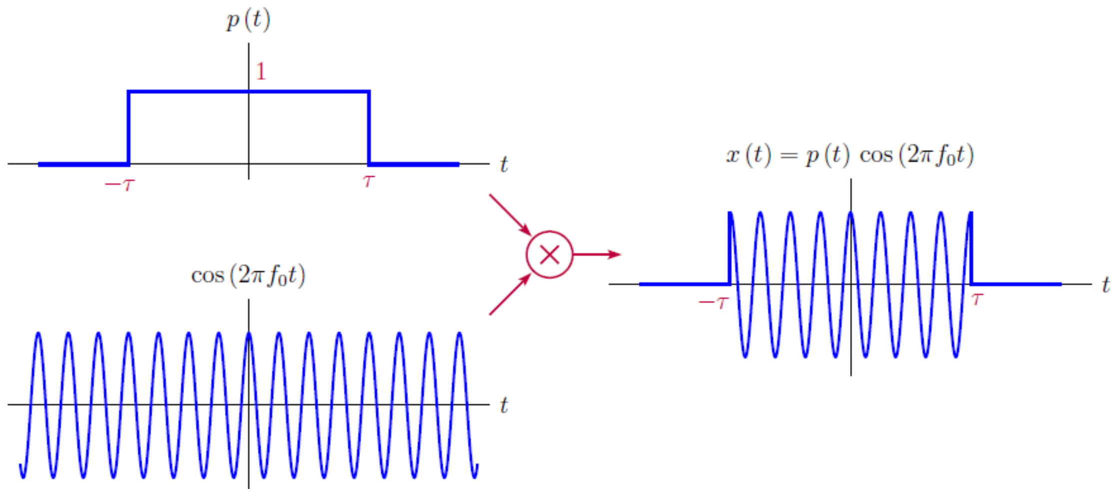
Solution:

Let $p(t)$ be defined as

$$p(t) = \Pi\left(\frac{t}{2\tau}\right)$$

Express $x(t)$ using $p(t)$:

$$x(t) = p(t)\cos(2\pi f_0 t)$$



4.3 Analysis of Non-Periodic Continuous-Time Signals

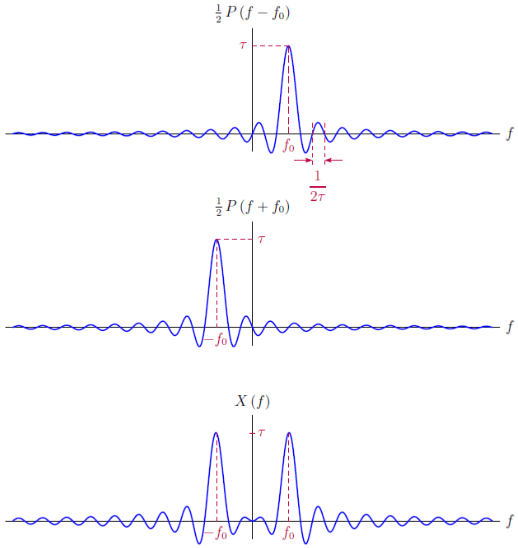
Example 4.29 (continued)

The transform of the pulse $p\left(t\right)$ is

$$P\left(f\right)=\mathcal{F}\left\{p\left(t\right)\right\}=2\tau\;\text{sinc}\left(2\tau f\right)$$

Apply the modulation property:

$$\begin{aligned}X\left(\omega\right)=&\frac{1}{2}\,P\left(f-f_0\right)+\frac{1}{2}\,P\left(f+f_0\right)\\=&\tau\;\text{sinc}\left(2\tau\left(f+f_0\right)\right)+\tau\;\text{sinc}\left(2\tau\left(f-f_0\right)\right)\end{aligned}$$



4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Time and frequency scaling:

Time and frequency scaling property

For a transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

it can be shown that

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

The parameter a is any nonzero and real-valued constant.

4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Differentiation in the time domain:

Differentiation in time property

For a given transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

it can be shown that

$$\frac{d^n}{dt^n}[x(t)] \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^n X(\omega)$$

If we choose to use f instead or ω, then

$$\frac{d^n}{dt^n}[x(t)] \overset{\mathcal{F}}{\longleftrightarrow} (j2\pi f)^n X(f)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Differentiation in the frequency domain:

Differentiation in frequency property

For a transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \tag{1}$$

it can be shown that

$$(-jt)^n \, x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{d^n}{d\omega^n}\left[X(\omega)\right]$$

If we choose to use f instead or ω, then

$$(-j2\pi t)^n \, x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{d^n}{df^n}\left[X(f)\right]$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Multiplication of two signals:

Multiplication property

For two transform pairs

$$x_1(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

it can be shown that

$$x_1(t)\, x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

If we choose to use f instead or ω, then

$$x_1(t)\, x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(f) * X_2(f)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Example 4.34

Transform of a truncated sinusoidal signal

A sinusoidal signal that is time-limited in the interval $-\tau < t < \tau$ is given by

$$x(t) = \begin{cases} \cos(2\pi f_0 t) \,, & -\tau < t < \tau \\ 0 \,, & \text{otherwise} \end{cases}$$

Determine the Fourier transform of this signal using the multiplication property.

Solution:

Let $x_1(t)$ and $x_2(t)$ be defined as

$$x_1(t) = \cos(2\pi f_0 t) \quad \text{and} \quad x_2(t) = \Pi\left(\frac{t}{2\tau}\right)$$

so that

$$x(t) = x_1(t)\, x_2(t) = \cos(2\pi f_0 t)\, \Pi\left(\frac{t}{2\tau}\right)$$

$$X_1(f) = \frac{1}{2}\,\delta(f+f_0) + \frac{1}{2}\,\delta(f-f_0) \qquad \text{and} \qquad X_2(f) = 2\tau\, \text{sinc}(2\tau f)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

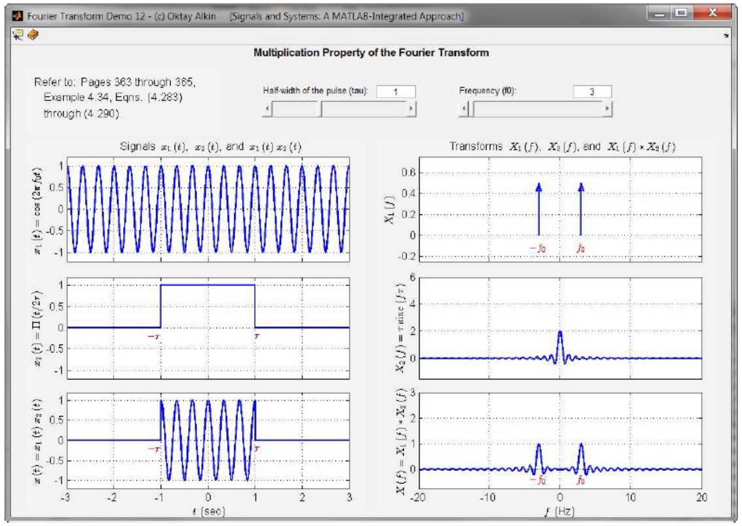
Example 4.34 (continued)

Use multiplication property:

$$
\begin{aligned}
X(f) &= X_1(f) * X_2(f) \\
&= \tau \ \mathrm{sinc}\left(2\tau\,(f+f_0)\right) + \tau \ \mathrm{sinc}\left(2\tau\,(f-f_0)\right)
\end{aligned}
$$

Interactive demo: ft_demo12.m

Explore the multiplication property of the Fourier transform. Change values of parameters τ and f_0 and observe the effects on the signal and its transform.



4.3 Analysis of Non-Periodic Continuous-Time Signals

Properties of the Fourier transform (continued)

Integration:

Integration property

For a transform pair

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

it can be shown that

$$\int_{-\infty}^{t} x(\lambda)\,d\lambda \xleftrightarrow{\mathcal{F}} \frac{X(\omega)}{j\omega} + \pi\,X(0)\,\delta(\omega)$$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Property	Signal	Transform
Linearity	$\alpha\, x_1(t) + \beta\, x_2(t)$	$\alpha\, X_1(\omega) + \beta\, X_2(\omega)$
Duality	$X(t)$	$2\pi\, x(-\omega)$
Conjugate symmetry	$x(t)$ real	$X^*(\omega) = X(-\omega)$ Magnitude: $\lvert X(-\omega)\rvert = \lvert X(\omega)\rvert$ Phase: $\Theta(-\omega) = -\Theta(\omega)$ Real part: $X_r(-\omega) = X_r(\omega)$ Imaginary part: $X_i(-\omega) = -X_i(\omega)$
Conjugate antisymmetry	$x(t)$ imaginary	$X^*(\omega) = -X(-\omega)$ Magnitude: $\lvert X(-\omega)\rvert = \lvert X(\omega)\rvert$ Phase: $\Theta(-\omega) = -\Theta(\omega) \mp \pi$ Real part: $X_r(-\omega) = -X_r(\omega)$ Imaginary part: $X_i(-\omega) = X_i(\omega)$
Even signal	$x(-t) = x(t)$	$\text{Im}\{X(\omega)\} = 0$
Odd signal	$x(-t) = -x(t)$	$\text{Re}\{X(\omega)\} = 0$
Time shifting	$x(t-\tau)$	$X(\omega)\, e^{-j\omega\tau}$
Frequency shifting	$x(t)\, e^{j\omega_0 t}$	$X(\omega - \omega_0)$
Modulation property	$x(t)\, \cos(\omega_0 t)$	$\frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$
Time and frequency scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$
Differentiation in time	$\frac{d^n}{dt^n}[x(t)]$	$(j\omega)^n\, X(\omega)$
Differentiation in frequency	$(-jt)^n\, x(t)$	$\frac{d^n}{d\omega^n}[X(\omega)]$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)\, X_2(\omega)$
Multiplication	$x_1(t)\, x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Integration	$\int_{-\infty}^{t} x(\lambda)\, d\lambda$	$\frac{X(\omega)}{j\omega} + \pi\, X(0)\, \delta(\omega)$

4.3 Analysis of Non-Periodic Continuous-Time Signals

Parseval's theorem

Parseval's theorem

For a periodic power signal $\tilde{x}(t)$ with period of T_0 and EFS coefficients $\{c_k\}$ it can be shown that

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0}|\tilde{x}(t)|^2\,dt=\sum_{k=-\infty}^{\infty}|c_k|^2$$

For a non-periodic energy signal $x(t)$ with a Fourier transform $X(f)$, the following holds true:

$$\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_{-\infty}^{\infty}|X(f)|^2\,df$$

4.4 Energy and Power in the Frequency Domain

Energy and power spectral density

Power spectral density for a periodic signal

$$S_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \, \delta(f - kf_0)$$

$$S_x(\omega) = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2 \, \delta(\omega - k\omega_0)$$

Compute the normalized average power of $\tilde{x}(t)$ that is within a specific frequency range $(-f_0, f_0)$:

$$P_x \text{ in } (-f_0, f_0) = \int_{-f_0}^{f_0} S_x(f) \, df = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} S_x(\omega) \, d\omega$$

4.4 Energy and Power in the Frequency Domain

Energy and power spectral density

Energy spectral density for a non-periodic signal

$$G_x(f) = |X(f)|^2$$

$$G_x(\omega) = |X(\omega)|^2$$

Compute the normalized average energy of $x(t)$ that is within a specific frequency range $(-f_0, f_0)$:

$$E_x \text{ in } (-f_0, f_0) = \int_{-f_0}^{f_0} G_x(f)\, df = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} G_x(\omega)\, d\omega$$

4.4 Energy and Power in the Frequency Domain

Energy and power spectral density (continued)

Some non-periodic signals are power signals, therefore their energy cannot be computed. One example of such a signal is the unit-step function. The normalized average power in a non-periodic signal is

$$P_x = \lim_{T\to\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \; dt \right]$$

Power spectral density for a non-periodic power signal

$$S_x\left(f\right) = \lim_{T\to\infty} \left[\frac{1}{T} \left| X_T\left(f\right) \right|^2 \right]$$

where

$$x_T\left(t\right) = \begin{cases} x\left(t\right) \; , & -T/2 < t < T/2 \\ 0 \; , & \text{otherwise} \end{cases}$$

and

$$X_T\left(f\right) = \mathcal{F}\left\{ x_T\left(t\right) \right\} = \int_{-T/2}^{T/2} x_T\left(t\right) \, e^{-j2\pi f t} \, dt$$

4.4 Energy and Power in the Frequency Domain

Energy and power spectral density

Energy spectral density for a non-periodic signal

$$G_x(f) = |X(f)|^2$$

$$G_x(\omega) = |X(\omega)|^2$$

Compute the normalized average energy of $x(t)$ that is within a specific frequency range $(-f_0, f_0)$:

$$E_x \text{ in } (-f_0, f_0) = \int_{-f_0}^{f_0} G_x(f)\, df = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} G_x(\omega)\, d\omega$$

4.4 Energy and Power in the Frequency Domain

Example 4.39

Power spectral density of a sinusoidal signal

Find the power spectral density of the signal $\tilde{x}(t) = 5\cos(200\pi t)$.

Solution:

Use Euler's formula:

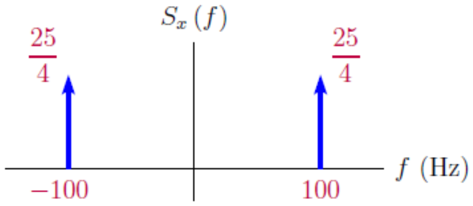
$$x(t) = \frac{5}{2}\,e^{-j200\pi t} + \frac{5}{2}\,e^{j200\pi t}$$

EFS coefficients:

$$c_{-1} = c_1 = \frac{5}{2}$$

The fundamental frequency is $f_0 = 100$ Hz. The power spectral density is

$$S_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2\,\delta(f - 100n) = \frac{25}{4}\,\delta(f + 100) + \frac{25}{4}\,\delta(f - 100)$$



4.4 Energy and Power in the Frequency Domain

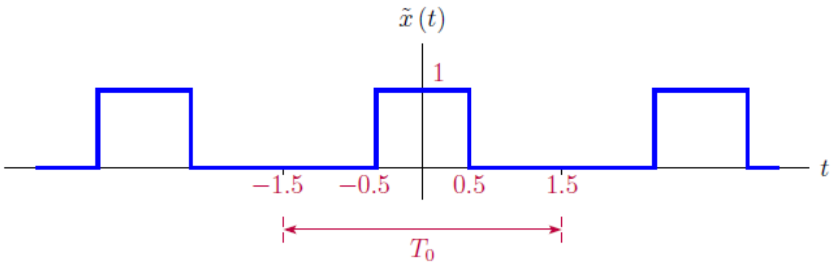
Example 4.38

Power spectral density of a periodic pulse train

The EFS coefficients for the periodic pulse train $\tilde{x}(t)$ are

$$c_k = \frac{1}{3}\,\text{sinc}\,(k/3)$$

Determine the power spectral density for $x(t)$. Also find the total power, the dc power, the power in the first three harmonics, and the power above 1 Hz.



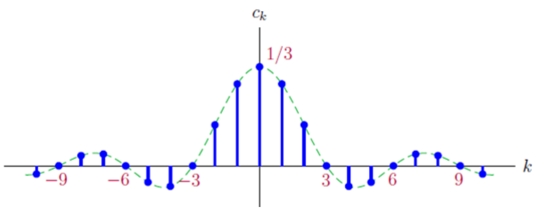
Solution: The period of the signal is $T_0 = 3$ s, and therefore the fundamental frequency is $f_0 = \frac{1}{3}$ Hz. The power spectral density is

$$S_x(f) = \sum_{k=-\infty}^{\infty} \left| \frac{1}{3}\,\text{sinc}\,(k/3) \right|^2 \delta(f - k/3)$$

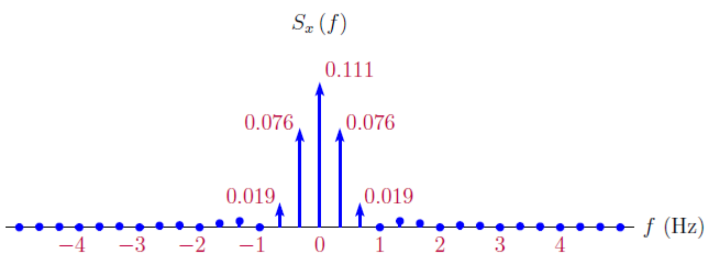
4.4 Energy and Power in the Frequency Domain

Example 4.5

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \left(\frac{\sin(\pi k/3)}{\pi k} \right) e^{j2\pi kt/3}$$



Example 4.38 (continued)



The total power in the signal $x(t)$:

$$P_x = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |\tilde{x}(t)|^2 \, dt = \frac{1}{3} \int_{-0.5}^{0.5} (1)^2 \, dt = 0.3333$$

The dc power in the signal $x(t)$:

$$P_{dc} = |c_0|^2 = (0.3333)^2 = 0.1111$$

The power in the fundamental frequency:

$$P_1 = |c_{-1}|^2 + |c_1|^2 = 2\,|c_1|^2 = 2\,(0.0760) = 0.1520$$

4.5 System Function Concept

System function concept

System function

The system function is the Fourier transform of the impulse response:

$$H(\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

In general, $H(\omega)$ is a complex function of ω, and can be written in polar form as

$$H(\omega) = |H(\omega)|\, e^{j\Theta(\omega)}$$

4.5 System Function Concept

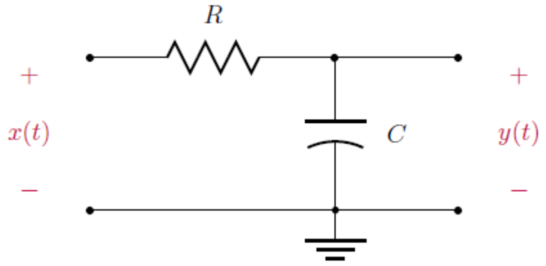
Example 4.43

System function for the simple RC circuit

The impulse response of the RC circuit shown was found in Example 2.18 to be

$$h(t) = \frac{1}{RC}\, e^{-t/RC}\, u(t)$$

Determine the system function.



Solution:

Take the Fourier transform of the impulse response:

$$H(\omega) = \int_0^{\infty} \frac{1}{RC}\, e^{-t/RC}\, e^{-j\omega t}\, dt = \frac{1}{1 + j\omega RC}$$

To simplify notation, define $\omega_c = 1/RC$:

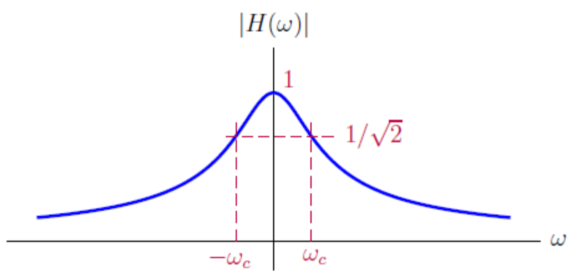
$$H(\omega) = \frac{1}{1 + j\,(\omega/\omega_c)}$$

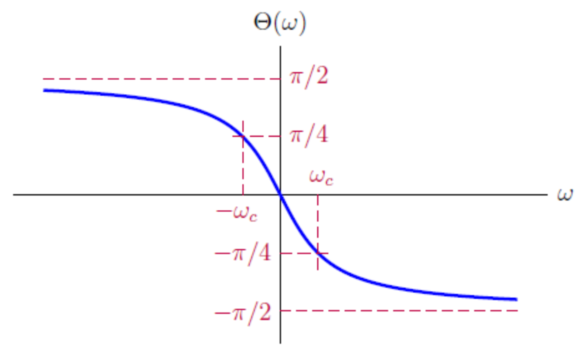
4.5 System Function Concept

Example 4.43 (continued)

The magnitude and the phase of the system function are

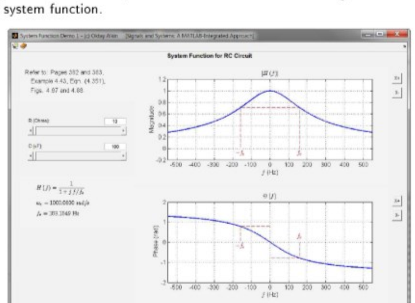
$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_c)^2}} \qquad \text{and} \qquad \Theta(\omega) = \measuredangle H(\omega) = -\tan^{-1}(\omega/\omega_c)$$





Interactive demo: sf_demo1.m

Vary circuit parameters R and C, and observe the effects on the magnitude and the phase of the system function.



The value of the system function at the frequency ω_c is $H(\omega_c) = 1/(1+j)$, and the corresponding magnitude is $|H(\omega_c)| = 1/\sqrt{2}$. Thus, ω_c represents the frequency at which the magnitude of the system function is 3 decibels below its peak value at $\omega = 0$, that is,

$$20\,\log_{10}\left[\frac{|H(\omega_c)|}{|H(0)|}\right] = 20\,\log_{10}\left[\frac{1}{\sqrt{2}}\right] \approx -3\text{ dB}$$

The frequency ω_c is often referred to as the *3-dB cutoff frequency* of the system.

Software resources:

4.5 System Function Concept

System function concept (continued)

Obtaining the system function from the differential equation:

1. Using the time differentiation property, write the Fourier transform of each term in the differential equation:

$$\frac{d^k y(t)}{dt^k} \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^k \, Y(\omega) \qquad k = 0, 1, \ldots$$

$$\frac{d^k x(t)}{dt^k} \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^k \, X(\omega) \qquad k = 0, 1, \ldots$$

2. Compute the system function as the ratio of the output transform to the input transform:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

4.5 System Function Concept

Example 4.44

Finding the system function from the differential equation

Determine the system function for a CTLTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\,\frac{dy(t)}{dt} + 26\,y(t) = x(t)$$

Solution:

Take the Fourier transform of both sides of the differential equation:

$$(j\omega)^2\; Y(\omega) + 2\;(j\omega)\;Y(\omega) + 26\,Y(\omega) = X(\omega)$$

$$\left[\left(26 - \omega^2\right) + j2\omega\right]\;Y(\omega) = X(\omega)$$

The system function is

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(26 - \omega^2) + j2\omega}$$

4.6 CTLTI Systems with Periodic Input Signals

CTLTI systems with periodic input signals

Periodic signal representation using TFS:

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

Periodic signal representation using EFS:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_0 t}$$

If a periodic signal is used as input to a CTLTI system, the superposition property can be utilized for finding the output signal.

4.6 CTLTI Systems with Periodic Input Signals

Response of a CTLTI system to periodic input signal

Let $\tilde{x}(t)$ be a signal that satisfies the existence conditions for the Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_0 t}$$

If $\tilde{x}(t)$ is used as the input signal to a CTLTI system, the response of the system is

$$\mathrm{Sys}\{\tilde{x}(t)\} = \sum_{k=-\infty}^{\infty} c_k \, H(k\omega_0) \, e^{jk\omega_0 t}$$

Important observations:

- For a CTLTI system driven by a periodic input signal, the output signal is also periodic with the same period.
- If the EFS coefficients of the input signal are $\{c_k \,;\, k = 1, \ldots, \infty\}$ then the EFS coefficients of the output signal are $\{c_k \, H(k\omega_0) \,;\, k = 1, \ldots, \infty\}$.

4.7 CTLTI Systems with Non-Periodic Input Signals

CTLTI systems with non-periodic input signals

Signal-system interaction

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

Assume that

- the system is stable ensuring that $H(\omega)$ converges, and
- the input signal has a Fourier transform.

$$Y(\omega) = H(\omega)\, X(\omega)$$

$$|Y(\omega)| = |H(\omega)|\, |X(\omega)|$$

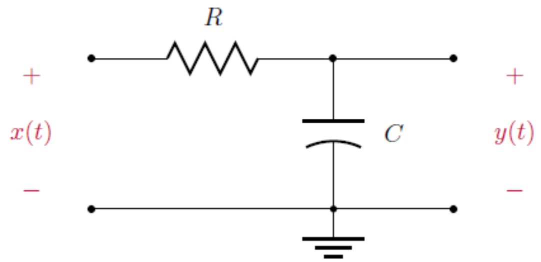
$$\measuredangle Y(\omega) = \measuredangle X(\omega) + \Theta(\omega)$$

4.7 CTLTI Systems with Non-Periodic Input Signals

Example 4.47

Pulse response of RC circuit revisited

Consider the RC circuit shown. Let $f_c = 1/RC = 80$ Hz. Determine the Fourier transform of the response of the system to the unit pulse input signal $x(t) = \Pi(t)$.



Solution:

The system function of the RC circuit was found in Example 4.43 to be

$$H(f) = \frac{1}{1 + j\,(f/f_c)}$$

The transform of the input signal is

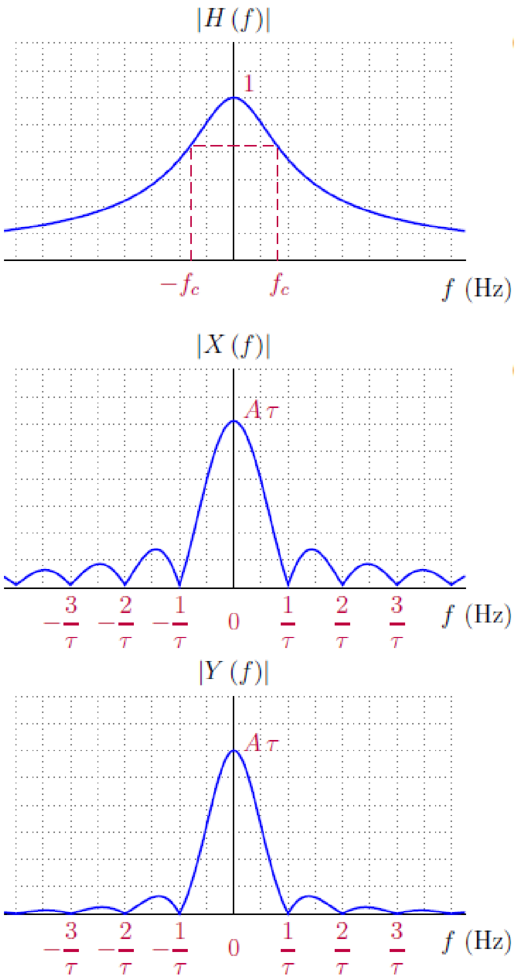
$$X(f) = \text{sinc}(f)$$

Using $f_c = 80$ Hz, the transform of the output signal is

$$Y(f) = H(f)\,X(f) = \frac{1}{1 + j\left(\frac{f}{80}\right)}\,\text{sinc}(f)$$

4.7 CTLTI Systems with Non-Periodic Input Signals

$$Y(f) = H(f)\,X(f) = \frac{1}{1 + j\left(\frac{f}{80}\right)}\,\text{sinc}(f)$$



Example 4.47 (continued)

Magnitude of the output transform:

$$|Y(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{80}\right)^2}}\,|\text{sinc}(f)|$$

Phase of the output transform

$$\measuredangle Y(f) = -\tan^{-1}\left(\frac{f}{80}\right) + \measuredangle\left[\text{sinc}(f)\right]$$